Stochastic approximation – a method for solving a problem of competing antagonism

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Abstract

Stochastic approximation is a relatively new technique for studying the properties of an experimental case, having manifold applications in different fields of activity, such as: medical field, biology, agriculture, engineering, economics, etc. In this paper, we discuss about an application of stochastic approximation from the viticulture field. We are interested by to determine the optimal quantities of two factors: Atonik and Ronilan, so the main shoots reach a well-balanced growth obviously influencing on the grape yield. We try to solve this problem, applying the competing antagonism technique.

Keywords: stochastic approximation, random value, repartition function, mean, dispersion.

Introduction

For a better understanding of what stochastic approximation implies, let us take into consideration the following examples:

(a) The specialists in agriculture encounter the problem of determining a critical dosage when testing insecticides, in other words, which is the optimal amount of insecticides that should be applied in case of a certain crop, so that it might be efficient without having a bad influence on the final production.

(b) A problem in the same field consists in establishing the amount *x* of fertilisers used for fertilising an experimental plot. The amount of cereals produced $Y(x)$, is a random variable which depends on *x* (where *x* must range in between certain limits). The yield will be different each year, even if the same amount of fertilisers *x* is kept (as far as other factors are also implied – e.g. rainfalls, agricultural works, etc.)

A general problem of stochastic approximation, as it is rendered in the research works belonging to M.T. Wassan [5], may be formulated as it follows:

We consider an experiment and a variable parameter *x* which the results of that experiment depend on; a random value x_1 for *x* is chosen, which leads to the value $y(x_1)$ of the random variable parameter $Y(x)$, this representing the results of the experiment; subsequently, *x* is assigned the value x_2 which determines the value $y(x_2)$, and the procedure may further go on.

By assignment, $F(x) = EY(x)$, *E* representing the mean of the random variable parameter $Y(x)$.

If a not growing succession of positive numbers $a_n, (a_n \geq a_{n+1}, (\forall) n \in N)$ is chosen – for example: $a_n = \frac{c}{n}$, where $c > 0$ has a random value, the problem under discussion is to determine a value x^* so that $F(x^*) = \alpha$.

Therefore, a recurrent relationship is established in order to select other *x* values for the next experiment:

$$x_{n+1} = x_n - \frac{c}{n}\left[y(x_n) - \alpha\right]. \qquad (1)$$

It is supposed that the n^{th} experiment was accomplished and a result was obtained, hence x_n and $y(x_n)$ values are known. Therefore, using the recurrent relation (1), *x* value used in the $n+1^{st}$ experiment may be determined.

If in the relationship (1), for its simplification, $\alpha = 0$ is chosen, this becomes:

$$x_{n+1} = x_n - \frac{c}{n} y(x_n) \quad (2)$$

It is noticeable that if $y(x_n)>0$ and taking into account the selection of c, it results $x_{n+1} < x_n$; and if $y(x_n) < 0$, then $x_{n+1} > x_n$. These observations are important for solving the equation:

$$F(x^*) = \alpha.$$

The conclusion of those above mentioned would be that: if $y(x_n)$ is positive, then x value for n+1 stage of the experiment should decrease, and if $y(x_n)$ is negative, then the chosen x of the n+1st stage should decrease.

What further interests us are the conditions that the string $(x_n)_n$ converge in a square mean, with the probability 1, for x^* value.

We may conclude this introductory part by several remarks:

There are practical problems where the stochastic approximation may be applied but those might be resolved also by using statistical methods or techniques of numerical analysis. The advantage of the stochastic approximation when compared to other methods consists in the fact that it is not necessary to know x value (that value which the experiment depends on) at the initial moment, but only its value at the final moment, that one being easily measured in practice. Moreover, it is not necessary to know the form of the regression function, or to estimate the unknown parameters; therefore, the stochastic approximation is a non-parametric technique that often generates a stochastic process, which is not a Markov process.

Another observation would be that the following aspects are important in a procedure of stochastic approximation:

- first of all we are interested in the convergence in itself and the convergence way of the sequence generated for finding out the solution of the equation $F(x^*) = \alpha$;
- on the other hand we would be interested to know the asymptotic repartition of the sequence;
- in the end, as far as the stochastic approximation is a sequential method, we are interested to know an optimum evidence for stopping the recurrent modus operandi, for a given situation.

Method

Further, we present the method of stochastic approximation applied to a problem belonging to the viticulture field of activity, and we use some practical dates given in [3].

The following genuine data come from the results of several analyses, which the Research & Development Institute for Viticulture and Enology, Valea Calugareasca made available for us.

For a better rooting and a successful joining of the grafting partners used in the grapevine grafting (a European grapevine susceptible to phylloxera was used as a scion; an American grapevine resistant to phylloxera was used as a rootstock), Atonik was used as a growth regulator. This product stimulates plant growth and fruit development in case of all the crops by activating the cytoplasm flow, by a quicker translocation of the raw sap and assimilates, ensuring a faster flow of the minerals and leading to an increase of the chlorophyll content. It also stimulates plant-rooting, seed and pollen germination, shoot growth and proliferation, flower fertility and fruit development; it stimulates the multiplication of micro-organisms in the soil, accelerates the decomposition of the organic matter, indirectly improving soil fertility. Atonik is different from the hormonal stimulators, avoiding the tissue over-saturation in water.

Immediately after using Atonik, the fungicide Ronilan 50 DE is used for controlling grapevine diseases. The former is used for growth stimulation, the latter for destroying different fungus-caused diseases (downy mildew, powdery mildew, grey rot).

Several grapevines were taken into study and each 5 days the growing process of the main shoots was registered after applying only Atonik in case of some plants and only Ronilan in case of the others. The following results were obtained:

Atonik dosage (mg / l)	**Ronilan dosage (mg / l)**	**Treated by Atonik (growth -cm)**	**Treated by Ronilan (growth -cm)**

5	5	3.3	2.3
5.5	4.5	4.6	5.0
5.3	4.7	9.8	7.4

As a conclusion, it is possible to apply a treatment to the grapevine either by using Atonik (factor 1), or by using Ronilan (factor 2).

The action of the first factor determines a certain answer which is measurable, namely a certain growth of the main shoots. Therefore, if x_1 is the variable quantity of Atonik applied to the grapevine by hectare, then $Y_1(x_1)$ is a random variable parameter, its values being measurable (rendered in the 3rd column of the above table) and representing the growth rates obtained after applying certain quantities of x_1, whereas under the action of Ronilan (i.e. x_2 factor), the growth rates obtained are given by the values of the random variable parameters $Y_2(x_2)$ (rendered in the 4th column of the above table).

By comparing the results representing the growth rates of the main shoots, it is noticeable that they are ampler when factor 1 gets involved; factor 2 must not be neglected, either. As it was already mentioned, this one is very important for controlling grapevine diseases.

What interests us is to determine the optimal quantities in case of both of the factors, so that the main shoots reach a well-balanced growth obviously influencing on the grape yield.

The previous situation is thought about by taking into account the model of the competing antagonism between the treatments, its mathematical formulation being further presented.

Result

We consider two families of random variable parameters $\{Y_1(x)\}$ and $\{Y_2(x)\}$ with the repartition function, mean and dispersion $F_i(Y(x)), M_i(x), \sigma_i^2(x), \quad i=1,2$.

In accordance with [1], we suppose that $M_1(x)$ and $M_2(x)$ enjoy the following properties,

(i) $M_i(x), i=1,2$ are strictly increasing and continual in *x*.

(ii) $(\forall)\alpha \in R$, given $(\exists)x_1^*$ and x_2^*, so that :

$$x_1^* + x_2^* = \alpha,$$

$$M_1(x_1^*) = M_2(x_2^*) \overset{not}{=} M(\alpha). \tag{4}$$

Let us consider a string $(a_n)_n$ of positive numbers having the following properties:

$$a_n > 0, \quad (\forall)n; \quad \sum_1^\infty a_n = \infty; \quad \sum_1^\infty a_n^2 < \infty. \tag{5}$$

If x_{11}, x_{21} are two random values representing the quantities initially applied of the two factors on two experimental objects, then, their effect (noticed after the measurement) is $Y_1(x_1)$, respectively $Y_2(x_2)$. At the next level determined by the values x_{12}, respectively x_{22}, the experimental performance is characterized by the relationship:

$$x_{i2} = x_{i1} + a_1\left[\overline{Y_1} - Y_i(x_{i1})\right], \quad i=1,2,$$

where $$\overline{Y_1} = \frac{Y_1(x_{11}) + Y_2(x_{21})}{2}.$$

We consider that x_{1n} and x_{2n} are the values which determine the experiment in case of the step *n*, and $Y_{1n} = Y_1(x_{1n})$, respectively $Y_{2n} = Y_2(x_{2n})$, are the values taken by the random variable parameters Y_1, Y_2 in the points x_{1n}, respectively x_{2n}, in other words, they represent the answer given by the experiment at both of the levels. We note the mean of the values observed at the step *n*, with $\overline{Y_n}$, namely:

$$\overline{Y_n} = \frac{Y_{1n} + Y_{2n}}{2}.$$

In this case, the levels x_{1n+1} and x_{2n+1} which determine the experiment at the step $n+1$, are given by the relationship:

$$x_{in+1} = x_{in} + a_n\left(\overline{Y_n} - Y_{in}\right), \quad i = 1,2, \quad n \geq 1. \tag{6}$$

We admit that for the given x_{1n}, respectively x_{2n}, the answers offered by the experiment Y_{1n}, Y_{2n} are independent. Let us consider after [4], also:

$$\sigma_i^2(x) < v < \infty, \quad i = 1,2, \quad (\forall)x, \tag{7}$$

And suppose that there are the constant values *A* and *B*, so that:

$$\left|M_i(x)\right| < A|x| + B < \infty, \quad i = 1,2, \quad (\forall)x. \tag{8}$$

Taking into account these conditions, in accordance with the theorem of Comer, concerning the convergence of a string of estimations, $(x_n)_n$ for a value x^*, in a square mean with the probability 1, it results that $(x_{1n})_n$ and $(x_{2n})_n$ defined by the relationship (6), converge in a square mean also with the probability 1 at L_1 and, respectively L_2 and $M_1\left(\lim_{n\to\infty} x_{1n}\right) = M_2\left(\lim_{n\to\infty} x_{2n}\right)$, with the probability 1. We can show, by using the method of reducing to the absurd, that $L_1 = x_1^*$ and $L_2 = x_2^*$.

For each $\alpha = x_{11} + x_{21}$ (addition of the values initially assigned to both of the factors, which the experiment depend on), we are able to define a family of random values (which establishes a correlation between the answer variable parameters $Y_1(x_1), Y_2(x_2)$):

$$V_\alpha(r) = \frac{Y_2\left(x_2^* + r\right) - Y_1\left(x_1^* - r\right)}{2}, \quad -\infty < r < \infty,$$

with $E\left\{V_\alpha(r)\right\}$ - strictly increasing function in *r*

$$E\left\{V_\alpha(0)\right\} = 0, \qquad \sigma_\alpha^2 = \frac{\sigma_2^2\left(x_2^* + r\right) + \sigma_1^2\left(x_1^* - r\right)}{4},$$

where $\sigma_\alpha^2(r)$ represents the dispersion of the variable parameter $V_\alpha(r)$.

Furthermore, in accordance with [2], we apply the technique of the real stochastic approximation in order to find the values x_1^* and x_2^*, each of them corresponding to the optimal dosage in case of the factor 1 and, respectively, factor 2.

Therefore, we build up the string $(r_n)_{n\geq 1}$, with r_1 - at random chosen and defined by the following recurrence relationship:

$$r_{n+1} = r_n - a_n v_\alpha(r_n), \tag{9}$$

Where $v_\alpha(r_n)$ is the answer observed at the level r_n.

For each set up α, the rule:

$$\begin{cases} x_{1n} = x_1^* - r_n \\ x_{2n} = x_2^* + r_n \end{cases},$$

establishes a correspondence between the strings $(x_{1n})_{n\geq 1}, (x_{2n})_{n\geq 1}$ defined by the relationship (6), and the string $(r_n)_{n\geq 1}$ defined by (9). Hence, the convergence of $(r_n)_n$ at 0, is equivalent with convergence of $(x_{1n})_n$ at x_1^* and of $(x_{2n})_n$ toward x_2^*.

We realize that the family of random variable parameters $\left(V_\alpha(r)\right)$ meets all the conditions of Dvoretzky theorem, therefore, we may conclude that whatever the initial values of the experiment might be, there are always x_1^* and x_2^* depending on those initial values, so that

$$P\left[\lim_{n\to\infty} E\left(x_{in} - x_i^*\right)^2 = 0\right] = 1, \quad i = 1, 2.$$

Then,

$$P\left[M_1\left(\lim_{n\to\infty} x_{1n}\right) = M_2\left(\lim_{n\to\infty} x_{2n}\right)\right] = 1,$$

results from the condition (ii).

Conclusions

In conclusion, as it is possible to establish the optimal doses by two factors, we can say that using the stochastic approximation has an important action on grapevine growth.
A generalization of the above rendered problem is that where, for a well-balanced development of the grapevines, more factors might be involved – it is the case of *multidimensional stochastic approximation*.

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